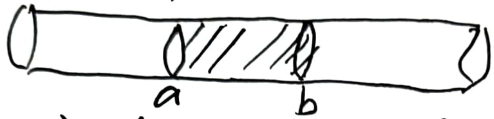
Lecture 4: Conservation Equations, Characteristics

- A conservation equation describes a system in which a quantity like energy is conserved, usually by setting up a pde for the density of that equation like the LaPlace Eqn.

Model Problem: Oxygen in Blood.

- Model an artery as a cylindrical tube



- let $u(t,x)$ denote oxygen concentration, in mass/length units

  total mass is then $m(t) = \int_a^b u(t,x)\,dx$ at time t

- instantaneous flow (an instant rate of change, like a derivative) is called flux. Denote it $q(t,x)$ in units mass/time.

  flux = concentration x velocity

- Assume velocity is independent of oxygen, so

  $q(t,x) = u(t,x)\,v(t,x)$ ~~...~~

- Conservation of mass means $m(t)$ changes only by the blood flowing in & out, or

$$\frac{dm}{dt}(t) = q(t,a) - q(t,b)$$

Let $q(t,\cdot)$ be continuously differentiable for all fixed t.

Then, $q(t,a) - q(t,b) = -\int_a^b \frac{\partial q}{\partial x}(t,x)\,dx$

By the Leibniz rule,

$$\frac{dm}{dt} = \int_a^b \frac{\partial u}{\partial t}\,dx \quad \text{if } u(t,x) \text{ is } C^1 \text{ in time.}$$

then, $\int_a^b \left(\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x}\right)dx = 0.$

we didn't specify a or b, so this must hold for all choices, giving that $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$

(if not, we can find a nonzero integral on some interval by continuity)

- This PDE $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$ is called the transport eqn.

- Since $q = uV$,

(D) $$\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} + u\frac{\partial V}{\partial x} = 0$$ $\rightsquigarrow$ a linear Conservation Eqn.

- This describes how we obtain PDEs from Conservation laws.

Lagrangian Derivatives + Characteristics.

- Motivated by (D), we investigate a general PDE of the form

(E) $\frac{\partial u}{\partial t} + V\frac{\partial u}{\partial x} + w = 0$ for $V = V(t,x)$, $w = w(t,x,u)$.

- We define a characteristic to be a trajectory $t \mapsto x(t)$ such that

$$\frac{dx}{dt}(t) = V(x,t)$$

~ if $V \in C^1$, Picard-Lindelöf shows that a unique solution to this exists in a nbhd. of each starting point (t_0, x_0)

ex.) Suppose $v(t,x) = at + b$ for $a, b \in \mathbb{R}$.
Then, $\dot{x}(t) = at + b \Rightarrow x(t) = \frac{a}{2}t^2 + bt + c$ for $c = x(0)$.

- Now, characteristics are quite a visual thing. The above characteristics give a family of parabolas. Since each characteristic is a 1D object, we may look at our concentration along the characteristic to reduce a PDE to an ODE. We denote this with

$$\frac{Du}{Dt} = \frac{d}{dt} u(t, x(t))$$

the "Lagrangian Derivative"

$\boxed{Th^{\underline{m}}}$ On each characteristic, the PDE (E) reduces to the ODE

$$\frac{Du}{Dt} + w(t, x(t), u(t, x(t))) = 0.$$

In particular, if $w \equiv 0$, then u is constant on the characteristics.

$\boxed{Pf}$ By the chain rule, $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = -w$ if u solves the PDE. $\square$

- If we can solve this ODE, we get a candidate for u. This is the method of characteristics.

e.g.) For $v(t,x) = at+b$, we have $\frac{\partial u}{\partial t} + (at+b)\frac{\partial u}{\partial x} = 0$ $(w=0)$ with initial condition $u(0,x) = g(x)$, for some $g \in C'(\mathbb{R})$.

Since $w=0$, u is constant on characteristics, so

$u(t, \frac{a}{2}t^2 + bt + c) = u(0,c) = g(c)$. To get a formula for $u(t,x)$, we set $x = \frac{a}{2}t^2 + bt + c$ to get $c = x - \frac{a}{2}t^2 - bt$ and $u(t,x) = g(x - \frac{a}{2}t^2 - bt)$

e.g.) Let $v(t,x) = a+bx$ for $x \geq 0$, $a, b > 0$. This corresponds to a shrinking velocity changing with position, such as diameter in a pipe.

Then, $\frac{dx}{dt} = a+bx$ gives $\frac{1}{b}\ln|a+bx| = t + C$

or $x(t) = \frac{1}{b}[Ke^{bt} - a]$ characteristic curves. $(K \in \mathbb{R})$.

- Since we restricted to $x \geq 0$, we index by t_0 so $x(t_0) = 0$,

or $x(t) = \frac{a}{b}[e^{b(t-t_0)} - 1]$

Characteristics

• With $v = a + bx$, the Conservation eqn. becomes

$$\frac{\partial u}{\partial t} + (a+bx)\frac{\partial u}{\partial x} + bu = 0$$

let us have boundary Condition $u(t,0) = f(t)$.

Then, by the thm,

$$\frac{Du}{Dt} + w = \frac{Du}{Dt} + bu = 0, \quad \text{giving} \quad u(t, x(t)) = Ae^{-bt}$$

to Solve for A, $u(t_0, 0) = f(t_0) = Ae^{-bt_0} \Rightarrow A = f(t_0)e^{bt_0}$

and $u(t, x(t)) = f(t_0)\, e^{bt_0} e^{-bt} = u\left(t, \frac{a}{b}\left[e^{b(t-t_0)} - 1\right]\right)$

then, $x = \frac{a}{b}\left[e^{b(t-t_0)} - 1\right] \Rightarrow t_0 = t + \frac{1}{b}\ln\left(\frac{a}{a+bx}\right)$

gives $u(t,x) = \left(\frac{a}{a+bx}\right) f\left(t + \frac{1}{b}\ln\left(\frac{a}{a+bx}\right)\right)$

General Method:

0.) Ensure the equation is of the form $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} + w = 0$

1.) Solve $\dot{x}(t) = v(t,x)$ at (t_0, x_0) to obtain Characteristic Curves $x(t)$

2.) Solve $\frac{Du}{Dt} + w = 0$ to get $u(t, x(t))$ with initial data

3.) Set $x = x(t)$ and invert to Solve for (t_0, x_0) in terms of x & t.

4.) Put this in $u(t, x(t))$ to get a formula $u(t,x)$.

• Note: there is a much more general method of Characteristics - See Evans. Ch. 3